If you have trouble viewing this, try the pdf of this post. You can download the code used to produce the figures in this post.

Beam hardening 2: the no-linearize theorem

The last post showed that beam hardening causes a nonlinearity between the log of the measurements and the A-vector. It is natural to think that we can eliminate the beam hardening artifacts by measuring the nonlinearity and then "linearizing" it with an inverse transformation. In this post, I will show that this is not possible in general. Although there are some special cases when we can linearize and a linearizing transformation may reduce the artifacts, we cannot do this for every object. I will show that this is due to the fact that we need at least a two dimension basis set to represent the attenuation coefficient.

The goal of linearizing

A conventional, non-energy selective CT system attempts to reconstruct the linear attenuation coefficient $\mu(\mathbf{r}, E_0)$ at a single average energy E_0, where $\mathbf{r}$ is the position in the cross section. Introducing the linear decomposition of the attenuation coefficient, $\mu(\mathbf{r}, E_0) = a_1(\mathbf{r})f_1(E_0) + a_2(\mathbf{r})f_2(E_0)$, we can see that by linearizing the CT system attempts to reconstruct a constant linear combination of the a_1 and a_2

$$\mu(\mathbf{r}, E_0) = a_1(\mathbf{r})f_1(E_0) + a_2(\mathbf{r})f_2(E_0) \tag{1}$$

where the $f_1(E_0)$ and $f_2(E_0)$ are constants. Integrating Eq. 1 along lines, this is equivalent to reconstructing

$$\mathcal{L} = A_1 f_1(E_0) + A_2 f_2(E_0). \tag{2}$$

From this, we see that the goal of linearizing is to compute a linear combination with constant coefficients of the A-vector components from a broad spectrum measurement.

The no-linearize theorem

I introduced and proved this theorem in Chapter 6 of my dissertation, which has other interesting results on beam hardening artifacts. Based on the goal of linearizing discussion in the previous section, we can state the no-linearize theorem as follows:

> Suppose we have a measurement I with a spectrum $S(E)$ and the object A-vector is $\left[\begin{array}{cc} A_1, & A_2 \end{array}\right]^T$ so that
>
> $$I(A_1, A_2) = \int S(E) \exp\left[-A_1 f_1(E) - A_2 f_2(E)\right] dE. \tag{3}$$
>
> If the object consists of more than one material and the spectrum is not monoenergetic, then an invertible function g does not exist such that
>
> $$g\left[I(A1, A2)\right] = A_1 f_1(E_0) + A_2 f_2(E_0) \tag{4}$$

where $f_1(E_0) = k_1$ and $f_2(E_0) = k_2$ are constants.

The proof has two steps. First, I will show that if g exists, then the contour lines of I in A-space, that is the set of points where it is a constant, are straight lines. Next, I show that except for special cases the contours are not straight lines. This completes the proof that g cannot exist by contradiction.

For the first part of the proof, we can reason as follows: The contour lines of I are defined by $I(A1, A2) = C$, where C is a constant. Applying g to both sides of this equation and using Eq. 4

$$g\left[I(A1, A2)\right] = k_1 A_1 + k_2 A_2 = g(C).$$

Since $g(C)$ is also a constant, if g exists then the contour curve is

$$k_1 A_1 + k_2 A_2 = constant. \tag{5}$$

This is the equation of a straight line, $A_2 = -{}^{k_1}/_{k_2} A_1 + constant$. This shows the first step of the proof.

For the second part of the proof, we need to show that the contours are not straight lines. Taking the total derivative of $I(A_1, A_2)$

$$dI = \frac{\partial I}{\partial A_1} dA_1 + \frac{\partial I}{\partial A_2} dA_2$$

On a contour of I, $dI = 0$ so solving for the slope

$$\frac{dA_2}{dA_1} = -\frac{\frac{\partial I}{\partial A_1}}{\frac{\partial I}{\partial A_2}}. \tag{6}$$

Differentiating Eq. 3 with respect to A_1 and A_2

$$\frac{\partial I}{\partial A_1} = -\int f_1(E) S(E) \exp\left[-A_1 f_1(E) - A_2 f_2(E)\right] dE$$

and

$$\frac{\partial I}{\partial A_2} = -\int f_2(E) S(E) \exp\left[-A_1 f_1(E) - A_2 f_2(E)\right] dE.$$

Substituting in Eq. 6

$$\frac{dA_2}{dA_1} = -\frac{\int f_1(E) S(E) \exp[-A_1 f_1(E) - A_2 f_2(E)] dE}{\int f_2(E) S(E) \exp[-A_1 f_1(E) - A_2 f_2(E)] dE}$$

Defining $u(E) = S(E) \exp\left[-A_1 f_1(E) - A_2 f_2(E)\right]$ and using bracket notation for the integrals

$$\frac{dA_2}{dA_1} = -\frac{\langle f_1, u\rangle}{\langle f_2, u\rangle}.$$

Multiplying across, the equation becomes

$$\langle f_1, u\rangle + \frac{dA_2}{dA_1} \langle f_2, u\rangle = 0. \tag{7}$$

By definition, the slope of a straight line is constant. Eq. 7 states that if the slope ${}^{dA_2}/_{dA_1}$ is constant, in the space of $u(E)$ functions, $f_1(E)$ and $f_2(E)$ are linearly dependent. Notice however that we have not said anything about the spectrum. Therefore the function u can be any function and Eq. 7 says that if the contours of $I(A_1, A_2)$ are straight lines then $f_1(E)$ and $f_2(E)$ are linearly dependent. But we know this is not true or else we would not need a two function basis set to represent $\mu(E)$. Therefore, the contours cannot be straight lines and this completes the proof of the no-linearize theorem by contradiction.

Linearizable cases

It turns out that the linearizable cases are the same as the cases for no beam hardening artifacts discussed in the previous post. The first case is a monoenergetic spectrum. Then

$$\begin{aligned} u(E_0) &= S(E_0) \exp\left[-A_1 f_1(E_0) - A_2 f_2(E_0)\right] \\ &= constant. \end{aligned}$$

In this case, the function space $u(E)$ is degenerate so the contour lines of $I(A_1, A_2)$ are straight lines even though $f_1(E)$ and $f_2(E)$ are linearly independent. In the second case, the object consists of a single material. Then the A-vector is constrained to be on a straight line through the origin so the possible A-space is not two dimensional and the contour lines are not meaningful.

Discussion

The no-linearize theorem does not say that we should not linearize. Linearizing does help to reduce artifacts. But it is not clear what material to use to define the linearizing curve. Is it the curve for, say, soft tissue or for bone? If the object composition does not conform to your choice then you will have artifacts so you cannot linearize perfectly for all possible materials. One of the advantages of energy-selective systems is that they can in principle eliminate beam hardening artifacts for any body material composition.

In the following posts, I will discuss another approach that is sometimes used to try to eliminate artifacts–using an iterative reconstruction algorithm to eliminate the “inconsistent” portions of the projections. That is, if you can eliminate the part of the projections that are due to the nonlinearity then the assumption is that you will eliminate beam hardening artifacts. However, I will show that nonlinear transformations of some objects are consistent so with these objects you will still have artifacts.

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References