If you have trouble viewing this, try the pdf of this post. You can download the code used to produce the figures in this post.

The constant covariance approximation to the CRLB with pileup

In my last post, I showed that the probability distribution of photon counting detector data with pileup is multivariate normal for the counts typically used in material selective imaging. With the normal distribution and a linear model, the Cramèr-Rao lower bound (CRLB) for the covariance of the A-vector data includes a term that depends on the change in the measurement data covariance with $\mathbf{A}$. Without pileup I show in this post and in the Appendix of my "Dimensionality and noise ..." paper[1], available for free download here, that the change in covariance term is negligible for large enough counts. In Appendix B of my "SNR with pileup ..." paper[2], I show that the term is also negligible with pileup. In this post, I will present and explain the code to reproduce the figures in that section.

Full CRLB for multivariate normal linear model

For a linear model $\delta\mathbf{L} = \mathbf{M}\delta\mathbf{A}$ with multivariate normal noise, Appendix 3.C of Kay[3] shows that the Fisher information matrix $\mathbf{F}(\mathbf{A})$ has elements

$$[\mathbf{F}(\mathbf{A})]_{ij} = \mathbf{M}(:,i)^T\mathbf{C}_\mathbf{L}^{-1}\mathbf{M}(:,j) + \frac{1}{2}tr\left[\mathbf{C}_\mathbf{L}^{-1}\frac{\partial\mathbf{C}_\mathbf{L}}{\partial A_i}\mathbf{C}_\mathbf{L}^{-1}\frac{\partial\mathbf{C}_\mathbf{L}}{\partial A_j}\right] \quad (1)$$

where $\mathbf{L} = -\log\left(^{I}/\mathbf{I_0}\right)$, $\mathbf{I}$ is the vector with elements the measurements with different spectra, $\mathbf{I_0}$ is the measurements vector with no object in the system, and $\mathbf{C_L}$ is the covariance of the logarithm of the measurements. The notation $tr\left[\right]$ means the trace of a matrix. The CRLB of the A-vector covariance $\mathbf{C_A}$is the inverse of $\mathbf{F}(\mathbf{A})$

$$\mathbf{C_A} = [\mathbf{F}(\mathbf{A})]^{-1}. \quad (2)$$

No pileup formulas

Using the linear decomposition of the attenuation coefficient

$$\mu(E) = a_1 f_1(E) + a_2 f_2(E)$$

the components of $\mathbf{I}$ with no pileup are

$$I_k(\mathbf{A}) = \int s_k(E)\, e^{-A_1 f_1(E) - A f_2(E)} dE, \quad (3)$$

where $s_k(E)$ is the k-th measurement spectrum. The k component of $\mathbf{L}$ is therefore

$$L_k = -\log\left(\frac{\int s_k(E)\, e^{-A_1 f_1(E) - A f_2(E)} dE}{\int s_k(E)\, dE}\right).$$

Differentiating the linear model, the elements of $\mathbf{M}$ are

$$M_{ij} = \frac{\partial L_i}{\partial A_j} = \frac{\int s_i(E)\, e^{-A_1 f_1(E) - A f_2(E)} f_j(E) dE}{\int s_i(E)\, e^{-A_1 f_1(E) - A f_2(E)} dE}$$

Noticing that

$$\hat{s}_i(E) = \frac{s_i(E)e^{-A_1 f_1(E) - A f_2(E)}}{\int s_i\,(E)\,e^{-A_1 f_1(E) - A f_2(E)} dE}$$

is the normalized transmitted spectrum for measurement i, M_{ij} is the effective value of the basis set function j in measurement spectrum i, $M_{ij} = \langle f_j \rangle_{\hat{s}_i}$. The complete matrix is then

$$\mathbf{M} = \frac{\partial \mathbf{L}(\mathbf{A})}{\partial \mathbf{A}} = \langle \mathbf{f} \rangle \tag{4}$$

For PHA and no pileup, the bin counts are independent and Poisson distributed so the covariance of the log measurement vector $\mathbf{L}$ is

$$\mathbf{C_L} = \begin{bmatrix} 1/I_1 & & \\ & \ddots & \\ & & 1/I_{nbins} \end{bmatrix}$$

Using this covariance, the derivatives in the second term of the Fisher matrix Eq. 1 are

$$\begin{aligned} \frac{\partial \mathbf{C_{L,kk}}}{\partial \mathbf{A_i}} &= \frac{\partial}{\partial \mathbf{A_i}}\left(\frac{1}{I_k}\right) \\ &= -\frac{1}{I_k^2}\frac{\partial I_k}{\partial A_i} \\ &= -\frac{M_{ki}}{I_k} \end{aligned} \tag{5}$$

Pileup formulas

With pileup, photons with different energies contribute to each measurement so Eq. 3 is no longer accurate. The gradient matrix is still defined as $\mathbf{M} = \frac{\partial \mathbf{L}}{\partial \mathbf{A}}$ so to compute it, we approximate the derivative from the first difference

$$\mathbf{M} \approx \frac{\mathbf{\Delta L}}{\mathbf{\Delta A}} \tag{6}$$

As stated in the paper

> To compute $\mathbf{\Delta L}$, we first compute the spectra through the object with attenuation $\mathbf{A}$ and then with $\mathbf{A}+\Delta\mathbf{A}$. The transmitted spectra are not affected by pileup since they occur before the measurement. These transmitted spectra are then used to compute the expected values of the measurements with pileup using the formulas in Section 2 of the paper.

This was implemented in the function $CRboundNoLinearizeWithPileup.m$. The relevant section of code is shown in the text box below from the subfunction $MLOC$:

```
for kdim = 1:specdat.nbasis
    sp2 = specdat;
    sp2.specnum = exp(-specdat.mus*dAs(kdim,:)').*specdat.specnum; % note transpose
    N2 = sum(sp2.specnum)*dE;
    rate2 = N2/Tintegrate;
    eta2 = rate2*deadtime;
    Nrec_bar2 = N2/(1 + eta2);
    sp_w_pileup2 = SpectrumWithPileup(sp2,eta2);
    fracs2 = BinFracsFromIndexes(sp_w_pileup2, specdat.idx_threshold);
    nrec_bins2 = Nrec_bar2*fracs2; % nbars for each bin
    M_NK(:,kdim) = -(log(nrec_bins2) - log(nrec_bins))/dAs(kdim,kdim);
end
```

The code first computes the PHA bin counts at the operating point in the first part of $MLOC$ (not shown). Then in the loop shown in the text box, it steps off as specified in the dAs matrix in each dimension, computes the new spectrum due to the additional dAs, computes the resulting spectrum with pileup, uses this to compute the new recorded bin counts, and then subtracts the logs of the new counts from the logs of the operating point counts and divides by the dAs component to give the first difference approximation to the derivative.

We can also approximate the derivatives in the second term of the Fisher matrix with pileup as the first difference.

$$\frac{\partial \mathbf{C_L}}{\partial A_i} \approx \frac{\Delta \mathbf{C_L}}{\Delta A_i} \tag{7}$$

The first difference of the covariance is computed in the $dRdALOC$ subfunction. This function is shown in the text box below:

```
function dRdA = dRdALOC(specdat, Tintegrate, deadtime,dAs)
nbins = numel(specdat.idx_threshold) + 1;
dRdA = zeros(nbins,nbins,specdat.nbasis); % nbasis planes, one for each A-vector component
cv0 = covLOC(specdat, Tintegrate, deadtime); % covariance at operating point
for kdim = 1:specdat.nbasis
    sp2 = specdat;
    sp2.specnum = exp(-specdat.mus*dAs(kdim,:)').*specdat.specnum; % transmitted spectrum -- note transpos
    cv = covLOC(sp2, Tintegrate, deadtime);
    dRdA(:,:,kdim) = (cv - cv0)/dAs(kdim,kdim);
end
```

The approach is the same as with the computation of $\mathbf{M}$. The covariance at the operating point $cv0$ is computed first using the $covLOC$ subfunction. The spectrum incident on the detector after stepping off in each dimension is used to compute the new covariance with pileup with $covLOC$. The first difference divided by the step size for that dimension is the approximation to the covariance derivative, $dRdA$. Notice that $dRdA$ is a three dimensional array.

The rest of $CRboundNoLinearizeWithPileup.m$ is a straightforward implementation of Eqs. 1 and 2. The inverse of the first term, F_1, is the constant covariance approximation and the inverse of $F_1 + F_2$ is the full CRLB.

Test pileup formulas

I tested the first difference approximation to the pileup formulas by comparing them with the actual derivatives for the zero dead time case. In this case, both formulas should

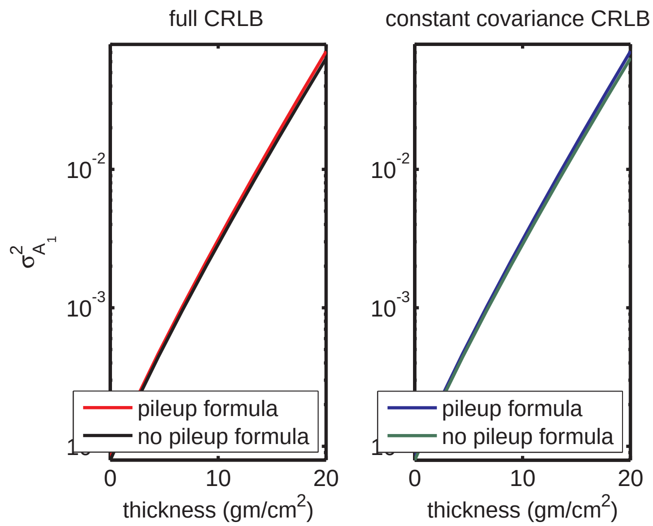

Figure 1: Compare $\mathbf{A_1}$ variance computed with no-pileup and pileup formulas. The plot shows the variance on a line through A-space as a function of the distance from the origin. The left plot compares the full variance while the right plot is for the constant covariance approximation. The pileup formulas with zero dead time using the first difference approximations of Eqs. 6 and 7 are compared with the actual derivatives in Eqs. 4 and 5.

give the same result. I did this for a set of A-vector values on a line through the origin in A-space. The results are shown in Fig. 1. The two functions' outputs are plotted versus distance from the origin in A-space. The left plot is for the full CRLB formula including the second term of the Fisher matrix while the right is only the first term. Notice that the first difference function is very close to the actual derivative function for both the constant covariance and full CRLB. Also notice that the left and right plots are indistinguishable. This is because, as will be shown in the next section, the constant covariance CRLB is nearly equal to the complete CRLB.

Constant covariance CRLB error with pileup

We can define the fractional error as

$$frac.err. = \frac{\|\mathbf{C}_{\mathbf{A},\mathbf{CRLB}} - \mathbf{C}_{\mathbf{A},\mathbf{CRLB},\mathbf{const\ cov}}\|}{\|\mathbf{C}_{\mathbf{A},\mathbf{CRLB}}\|} \tag{8}$$

where the symbol $\|\ \|$ denotes a matrix norm. Fig. 2 shows the error as a function of the number of photons for five values of the pileup parameter, η photons per dead time. The top plot is for three bin PHA and the bottom is for the NQ detector. Notice that for a given number of photons, the error with PHA increases but the error with the NQ detector decreases as η increases.

Conclusion

Both with and without pileup, the constant covariance approximation to the CRLB is accurate for photon counts less than those used in material selective applications.

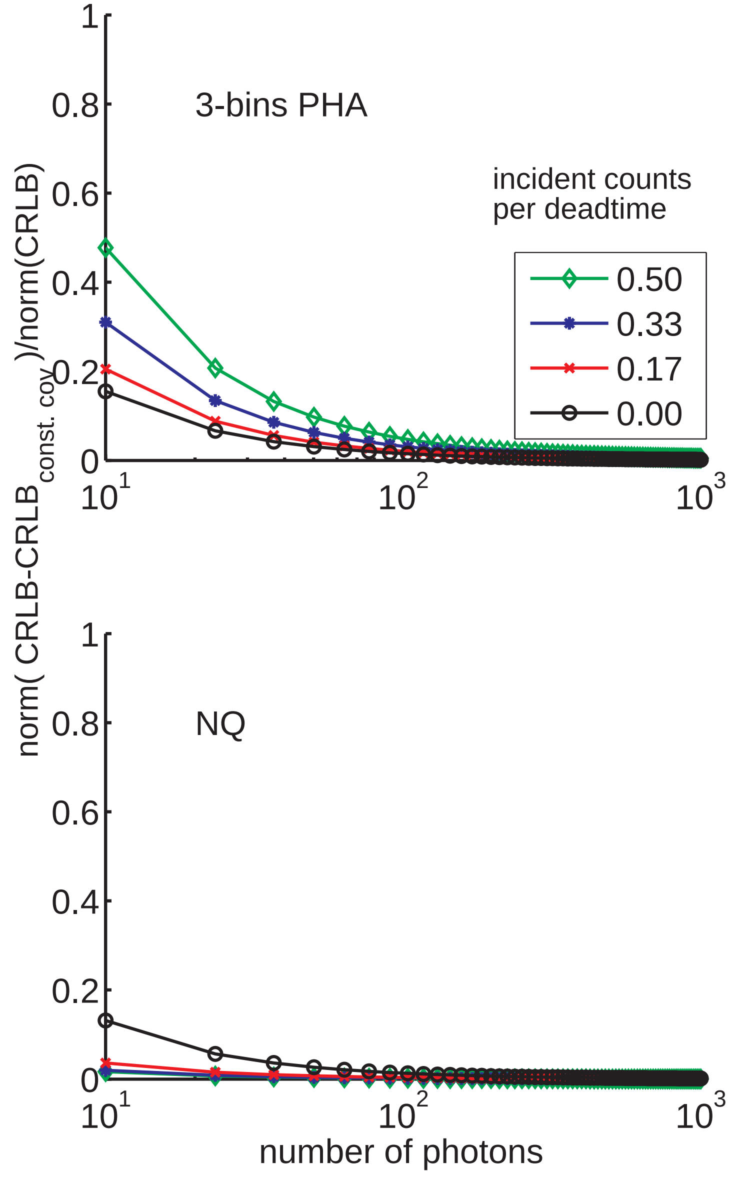

Figure 2: Constant covariance CRLB error with pileup. The top plot is for three bin PHA and the bottom is for the NQ detector.

–Bob Alvarez

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References

[1] R. E. Alvarez, "Dimensionality and noise in energy selective x-ray imaging," *Medical physics* **40**, no. 11, 111909, (2013).

[2] R. E. Alvarez, "Signal to noise ratio of energy selective x-ray photon counting systems with pileup," *Med. Phys.* **41**, no. 11, –, (2014).

[3] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory* , Upper Saddle River, NJ: Prentice Hall PTR, 1993.